

SUSY transformation of the Green function and a trace formula

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2005 J. Phys. A: Math. Gen. 38 7557

(<http://iopscience.iop.org/0305-4470/38/34/009>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.92

The article was downloaded on 03/06/2010 at 03:54

Please note that [terms and conditions apply](#).

SUSY transformation of the Green function and a trace formula

B F Samsonov¹, C V Sukumar² and A M Pupasov¹

¹ Physics Department of Tomsk State University, 36 Lenin Avenue, 634050 Tomsk, Russia

² Wadham College, Oxford OX1 3PN, UK

E-mail: samsonov@phys.tsu.ru and c.sukumar1@physics.oxford.ac.uk

Received 17 February 2005

Published 10 August 2005

Online at stacks.iop.org/JPhysA/38/7557

Abstract

An integral relation is established between the Green functions corresponding to two Hamiltonians which are supersymmetric (SUSY) partners and in general may possess both discrete and continuous spectra. It is shown that when the continuous spectrum is present, the trace of the difference of the Green functions for SUSY partners is a finite quantity which may or may not be equal to zero despite the divergence of the traces of each Green function. Our findings are illustrated by using the free particle example considered both on the whole real line and on a half line.

PACS numbers: 02.30.Gp, 03.65.Fd, 11.30.Pb

1. Introduction

At present, there is a growing interest in the study of different properties of transformations induced by supersymmetry (SUSY) in quantum mechanics. Recently, a special issue of this journal was devoted to research work in this subject. Despite the growing number of papers in this field, many questions still remain open and require further study. In particular, the authors are aware of only one paper [1] devoted to the study of SUSY transformations at the level of Green functions. For the case of a transformation deleting the ground state of the initial Hamiltonian, Sukumar has studied an integral relation between the Green functions for SUSY partners and has formulated conditions leading to the vanishing of some matrix elements of a Hamiltonian and related this property to a hidden supersymmetry of the system. Transformation of Green functions is not explicitly discussed in that paper. Moreover, we have found that formula (28) of [1] relating integrals over Green functions for SUSY partners may need to be corrected if a continuous spectrum is present.

In this paper, we give a simple formula for the Green function of the SUSY partner Hamiltonian both for confining and scattering potentials and generalize results of the paper [1] to the case where the continuous spectrum is present. As an application of this general

formula, we consider the case of the Schrödinger equation with a scattering potential defined both on the whole real axis and a half line when the Schrödinger equation is reduced to a singular Sturm–Liouville problem. A regular Sturm–Liouville problem is considered in a separate publication [2].

2. Green function of the Schrödinger equation

In this section, we cite some properties of the Green function of the one-dimensional Schrödinger equation for a spectral problem on the whole real line (see, e.g., [3, 4]) which are useful.

We consider the Schrödinger equation

$$(h_0 - E)\psi = 0 \quad h_0 = -d^2/dx^2 + V_0(x) \quad x \in (a, b) \quad (1)$$

supplemented by the boundary conditions $\psi(a) = \psi(b) = 0$. We will concentrate mostly on two cases; these are the whole real line $a = -\infty$ and $b = \infty$ and the half line $a = 0$ and $b = \infty$. We assume that the spectral set $\text{spec } h_0$ of this problem consists of M discrete points with the possibilities $M = 0$ or $M = \infty$ and possibly a continuum part filling the positive semiaxis.

The definition of the Green function used by different authors [3–5] may differ by a constant factor. We use a definition of the Green function represented as the kernel of the operator $(h_0 - E)^{-1}$ as an operator defined in the corresponding Hilbert space. It is well defined for all $E \notin \text{spec } h_0$. It has two different but equivalent representations. The first representation is obtained with the help of two real solutions of equation (1) with a fixed value of the parameter $E \notin \text{spec } h_0$, f_{l0} and f_{r0} ('left' and 'right' solutions), satisfying zero boundary conditions: $f_{l0}(a) = 0$, $f_{r0}(b) = 0$. Since they correspond to the same E , their Wronskian $W_0 = W(f_{r0}, f_{l0})$ does not depend on x and is a function of E only, and the Green function is

$$G_0(x, y, E) = f_{l0}(x, E)f_{r0}(y, E)/W_0 \quad x \leq y \quad (2)$$

$$G_0(y, x, E) = G_0(x, y, E). \quad (3)$$

These formulae are clearly equivalent to

$$G_0(x, y, E) = [f_{l0}(x, E)f_{r0}(y, E)\Theta(y - x) + f_{l0}(y, E)f_{r0}(x, E)\Theta(x - y)]/W_0 \quad (4)$$

where Θ is the Heaviside step function.

If the operator h_0 is essentially self-adjoint, the set of its discrete spectrum (if present) eigenfunctions $\{\psi_n\}$, $n = 0, 1, \dots, M$, $\langle \psi_n | \psi_m \rangle = \delta_{nm}$ together with the continuous spectrum eigenfunctions (also if present) ψ_k , $E = k^2 > 0$, $\langle \psi_k | \psi_{k'} \rangle = \delta(k - k')$, $\langle \psi_n | \psi_k \rangle = 0$, is complete in the Hilbert space

$$\sum_{n=0}^M \psi_n(x)\psi_n^*(y) + \int dk \psi_k(x)\psi_k^*(y) = \delta(x - y)$$

and the second representation of the Green function may be found in terms of this set as follows:

$$G_0(x, y, E) = \sum_{n=0}^M \frac{\psi_n(x)\psi_n^*(y)}{E_n - E} + \int \frac{\psi_k(x)\psi_k^*(y)}{k^2 - E} dk. \quad (5)$$

For the spectral problem on the whole real axis, the continuous spectrum is two-fold degenerate and the integrals over k run from minus infinity to infinity and for the problem on a half line they run from zero to infinity.

3. SUSY transformation of the Green function

It is well known (see, e.g., [6]) that there exist three kinds of SUSY transformations:

- (i) deleting the ground-state level of h_0 ,
- (ii) creating a new ground-state level and
- (iii) purely isospectral transformation.

In all cases, the partner Hamiltonian $h_1 = -d^2/dx^2 + V_1$ for h_0 is defined by the potential

$$V_1(x) = V_0(x) - 2w'(x) \quad w(x) = [\log u(x)]' \tag{6}$$

where u is a real solution to the initial equation $(h_0 - \alpha)u = 0$ with α known as the factorization constant. We adopt the notation that a derivative with respect to x is denoted by the prime symbol. To provide a nonsingular potential difference, α should be less than or equal to the ground-state energy of h_0 if it has a discrete spectrum or lower than the continuum threshold otherwise. The functions $\varphi_n = L\psi_n, n = 1, 2, \dots, M$ describe (unnormalized) bound states and $\varphi_E = L\psi_E$ correspond to (unnormalized) scattering states of h_1 . Here,

$$L = -d/dx + w(x) \tag{7}$$

is the transformation operator (intertwiner) satisfying $Lh_0 = h_1L$ and $Lu = 0$. The normalization constants are easily obtained with the help of the factorization property $L^+L = h_0 - \alpha$ where $L^+ = d/dx + w(x)$. The functions

$$\chi_n = (E_n - \alpha)^{-1/2}L\psi_n \quad \chi_E = (E - \alpha)^{-1/2}L\psi_E \tag{8}$$

form an orthonormal set.

Theorem 1. *Let $G_0(x, y, E)$ be the Green function for h_0 . Then for all three cases enumerated above the Green function for h_1 is*

$$G_1(x, y, E) = \frac{1}{E - \alpha} [L_x L_y G_0(x, y, E) - \delta(x - y)]. \tag{9}$$

In case (ii), it has a simple pole at $E = \alpha$. In cases (i) and (iii), it is regular at $E = \alpha$ and can be calculated as follows:

$$G_1(x, y, \alpha) = \left[L_x L_y \frac{\partial G_0(x, y, E)}{\partial E} \right]_{E=\alpha}. \tag{10}$$

Here, L_x is the operator given in (7) and L_y is the same operator where x is replaced by y .

Proof. In case (i), $u = \psi_0$ and the set $\{\chi_E, \chi_n, n = 1, 2, \dots, M\}$ is complete. Therefore,

$$G_1(x, y, E) = \sum_{n=1}^M \frac{\chi_n(x)\chi_n^*(y)}{E_n - E} + \int \frac{\chi_k(x)\chi_k^*(y)}{k^2 - E} dk. \tag{11}$$

Now we replace χ using (8) which yields

$$G_1(x, y, E) = \frac{1}{\alpha - E} L_x L_y \left(\sum_{n=1}^M \left[\frac{1}{E_n - \alpha} - \frac{1}{E_n - E} \right] \psi_n(x)\psi_n^*(y) + \int dk \left[\frac{1}{k^2 - \alpha} - \frac{1}{k^2 - E} \right] \psi_k(x)\psi_k^*(y) \right).$$

The statement for $E \neq \alpha$ follows from here if in the first sum and in the first integral we express $L\psi$ in terms of χ , make use of the completeness condition for the set χ and formula (5) for G_0 . The fact that here the sum starts from $n = 1$ and in (5) it starts from $n = 0$

cannot cause any problems since $L\psi_0 = 0$. For $E = \alpha$ formula (11) can be written in the form

$$G_1(x, y, \alpha) = \left[\frac{\partial}{\partial E} L_x L_y \left(\sum_{n=0}^M \frac{\psi_n(x)\psi_n^*(y)}{E_n - E} + \int \frac{\psi_k(x)\psi_k^*(y)}{k^2 - E} dk \right) \right]_{E=\alpha}$$

from which (10) follows in this case.

In case (ii), let $\chi_\alpha \sim 1/u$ be the normalized ground-state function of h_1 corresponding to the new discrete level $E = \alpha$. Then,

$$G_1(x, y, E) = \sum_{n=0}^M \frac{\chi_n(x)\chi_n^*(y)}{E_n - E} + \frac{\chi_\alpha(x)\chi_\alpha^*(y)}{\alpha - E} + \int \frac{\chi_k(x)\chi_k^*(y)}{k^2 - E} dk. \quad (12)$$

Now the use of exactly the same transformations as in case (i) reduces (12) to (9).

In case (iii), we start from the same formula (11) with the only difference that the sum now starts from $n = 0$ and following the same line of reasoning as in the earlier cases we get formula (9). It is interesting to note the intermediate result

$$G_1(x, y, E) = \frac{1}{\alpha - E} L_x L_y [G_0(x, y, \alpha) - G_0(x, y, E)] \quad (13)$$

which makes clear how formula (10) arises for this case by taking the limit $E \rightarrow \alpha$. The fact that in case (ii), the function (9) has a simple pole at $E = \alpha$ is a consequence of the equivalence between (12) and (9). \square

Corollary 1. *In terms of the special solutions f_{l0} and f_{r0} of the Schrödinger equation for h_0 the Green function G_1 for all three cases listed above may be expressed as follows:*

$$G_1(x, y, E) = \frac{1}{(E - \alpha)W_0} [\Theta(y - x)L_x f_{l0}(x, E)L_y f_{r0}(y, E) + \Theta(x - y)L_y f_{l0}(y, E)L_x f_{r0}(x, E)]. \quad (14)$$

In case (ii), this function has a simple pole at $E = \alpha$. In cases (i) and (iii), it is regular at $E = \alpha$ and can be calculated as follows:

$$G_1(x, y, \alpha) = \left[\frac{\partial}{\partial E} \frac{\Theta(y - x)L_x f_{l0}(x, E)L_y f_{r0}(y, E) + \Theta(x - y)L_y f_{l0}(y, E)L_x f_{r0}(x, E)}{W_0} \right]_{E=\alpha}. \quad (15)$$

To prove these formulae, we substitute G_0 as given in (4) into (9) and (10). Taking the derivative of the theta functions in (4) gives rise to the Dirac delta function which cancels out the delta function present in (9). Formula (14) is clearly valid since the SUSY transformations necessarily preserve the boundary conditions for all E except perhaps for $E = \alpha$. This implies that $f_{l1} = Lf_{l0}$ vanishes at $x = a$ and $f_{r1} = Lf_{r0}$ vanishes at $x = b$. The denominator in (14) is just the Wronskian of f_{r1} and f_{l1} and may be given in the form $W(f_{r1}, f_{l1}) = (E - \alpha)W(f_{r0}, f_{l0}) = (E - \alpha)W_0$.

4. Trace formulae

The trace of the Green function defined as [3] $\int_a^b G(x, x, E) dx$ is usually divergent if the system has a continuous spectrum. It is remarkable that the trace of the difference $G_0(x, x, E) - G_1(x, x, E)$ is a finite quantity which may or may not be equal to zero. In some cases, this fact may be explained by another remarkable property. It may happen that the

difference of infinite normalizations (they diverge as $\delta(x - y)$ when $y \rightarrow x$) of the continuous spectrum eigenfunctions of the two SUSY partners is a finite quantity.

Theorem 2. *Let $f_{l0}(x, E)$ and $f_{r0}(x, E)$ be solutions of the Schrödinger equation for h_0 satisfying the zero boundary conditions at the left and right bound of the interval (a, b) , respectively, and*

$$f_{l1}(x, E) = Lf_{l0}(x, E) \quad f_{r1}(x, E) = Lf_{r0}(x, E) \tag{16}$$

be similar solutions for h_1 related with h_0 by a SUSY transformation with α being the factorization constant. Let W_0 be the Wronskian of f_{r0} and f_{l0} , $W_0 = W(f_{r0}, f_{l0})$. Then,

$$\int_a^b [G_0(x, x, E) - G_1(x, x, E)] dx = \frac{Q(E)}{W_0(E - \alpha)} \tag{17}$$

where $Q(E)$ can be calculated by one of the following formulae:

$$Q(E) = (f_{r0}f_{l1})_{x=b} - (f_{r0}f_{l1})_{x=a} = (f_{l0}f_{r1})_{x=b} - (f_{l0}f_{r1})_{x=a} \tag{18}$$

$$= -W_0 + (f_{l0}f_{r1})_{x=b} - (f_{r0}f_{l1})_{x=a} = W_0 + (f_{r0}f_{l1})_{x=b} - (f_{l0}f_{r1})_{x=a}. \tag{19}$$

Proof. From corollary 1, it follows that $G_1(x, x, E) = \frac{1}{W_0(E - \alpha)} Lf_{l0}(x)Lf_{r0}(x)$. While integrating this expression over the interval (a, b) one can transfer the derivative present in L either from f_{l0} to f_{r0} or from f_{r0} to f_{l0} which leads to one of the following integrands $f_{l0}(x)L^+Lf_{r0}(x)$ or $f_{r0}(x)L^+Lf_{l0}(x)$. In both cases, the factorization property may be used to reduce the integrand to $(E - \alpha)f_{l0}(x)f_{r0}(x)$. Thus, we arrive at the relation

$$\int_a^b G_1(x, x) dx = \frac{1}{W_0} \int_a^b f_{l0}(x)f_{r0}(x) dx - \frac{Q(E)}{W_0(E - \alpha)} \tag{20}$$

where $Q(E)$ is given by (18). To prove (19), it is sufficient to note that

$$f_{l0}(x, E)f_{r1}(x, E) - f_{r0}(x, E)f_{l1}(x, E) = W_0 \tag{21}$$

which is a consequence of (16). The identification of the integrand on the right-hand side of (20) as $W_0G_0(x, x)$ then leads to the result given in (17). \square

Using the first of equalities (18), one can rewrite (17) as follows:

$$\int_a^b [G_0(x, x, E) - G_1(x, x, E)] dx = \frac{1}{\alpha - E} + \frac{(f_{l0}f_{r1})_{x=b} - (f_{r0}f_{l1})_{x=a}}{W_0(E - \alpha)}. \tag{22}$$

Now for case (i) where $\alpha = E_0$ if we compare this result with the corresponding difference which can be obtained directly from the expressions for G_0 given by (5) and for G_1 given by (11), the following feature may be noted: the first term on the right-hand side of (22) arises from the contribution to Green functions from the discrete spectra and the second term, which as we show below may be different from zero, is due to the presence of the continuous spectra. This contribution was neglected in [1]. So, theorem 2 presents a generalization of the result obtained in [1] to the case where a continuous spectrum may be present. As an application of this theorem, we are going to consider two particular cases of scattering potentials defined both on the whole real line and on a semiaxis.

Corollary 2. *If h_0 is a scattering Hamiltonian with the potential V_0 satisfying for the spectral problem on the whole line, the condition*

$$\int_{-\infty}^{\infty} (1 + |x|)|V_0(x)| dx < \infty$$

then for $E \neq \alpha$, $\text{Im} \sqrt{E} > 0$, the following equality

$$\int_{-\infty}^{\infty} [G_0(x, x, E) - G_1(x, x, E)] dx = \frac{\delta}{\kappa^2 + ia\kappa} - \frac{\delta}{\kappa^2 + a^2} \quad (23)$$

holds, where $E = \kappa^2$, $\alpha = -a^2$; $\delta = 1$ for case (i), $\delta = -1$ for case (ii) and $\delta = 0$ for case (iii).

Proof. The statement readily follows from the fact that any scattering potential has a pair of solutions (Jost solutions, see e.g., [7]) with the following asymptotics at the right infinity:

$$f_{l,r}(x, E) \rightarrow e^{\mp i\kappa x} \quad E = \kappa^2 \quad \text{Im} \kappa > 0 \quad x \rightarrow \infty$$

and similar asymptotics at the left infinity and the use of an appropriate part of equalities (18) and (19). The Wronskian W_0 for Jost solutions can easily be calculated, $W_0 = -2i\kappa$. \square

So, we see that despite the fact that for both h_0 and h_1 the continuous spectrum eigenfunctions are normalized to the Dirac delta function, i.e., that in both cases they have equal infinite norms, the difference of these infinities is a finite non-zero quantity in cases (i) and (ii) and it is zero in case (iii).

For instance, in case (ii) the following equality arises:

$$\int_{-\infty}^{\infty} \frac{P(k) dk}{k^2 - E} = R(E) \quad R(E) = \frac{-1}{\kappa^2 + ia\kappa} \quad E = \kappa^2 \quad \alpha = -a^2 \quad (24)$$

where

$$P(k) = \int_{-\infty}^{\infty} [|\psi_k(x)|^2 - |\chi_k(x)|^2] dx. \quad (25)$$

Equation (24) may be reduced to the Stieltjes transform and the function P may be found by the Stieltjes inversion formula (see, e.g., [4]). To establish this, we first note that the integral on the left-hand side of (24) is different from zero only if $P(k)$ is an even function which we assume to be the case. Therefore, it can be considered only for positive k s and we can let $k^2 = \lambda$. So, (24) takes the form

$$\int_{-\infty}^{\infty} \frac{d\rho(\lambda)}{\lambda - E} = R(E)$$

where the measure $\rho(\lambda)$ is continuous for $\lambda > 0$, $d\rho(\lambda) = \frac{1}{\sqrt{\lambda}} P(\lambda) d\lambda$ and such that for negative λ s the integral is zero. Now the Stieltjes inversion formula yields

$$\frac{P(\lambda)}{\sqrt{\lambda}} = \frac{\text{sign } \tau}{2\pi i} \lim_{\tau \rightarrow 0} [R(E) - R(\bar{E})] \quad E = \lambda + i\tau$$

where the bar over E denotes the complex conjugate to E . Note that because of the condition $\text{Im} \sqrt{E} > 0$ the square root of E has different signs for E in the upper and lower halves of the complex E -plane. Therefore, the function $R(E)$ has a cut along the real axis and the jump across this cut defines the function $P(\lambda)$. After a simple calculation one gets

$$P(\lambda) = a\pi^{-1}(\lambda^2 + a^2)^{-1}. \quad (26)$$

It must be noted that in the present case the interchange of the integrals over the space variable x taken in the difference of (5) and (11) at $y = x$ with the integral over the momentum k is justified.

Corollary 3. If h_0 is a scattering Hamiltonian with the potential V_0 for the spectral problem on a half line satisfying the condition

$$\int_0^{\infty} x |V_0(x)| dx < \infty,$$

then there exist only two kinds of SUSY transformations keeping the zero boundary condition at the origin. If h_0 has the discrete spectrum its ground state ψ_0 may be deleted ($u = \psi_0$, case (i)) and there is a possibility to keep the spectrum unchanged (case (iii)). The last possibility may be realized with $u(x) = f_l(x, E)$, $E < 0$ where f_l is such that $f_l(0, E) = 0$. In this case, the following trace formula is valid:

$$\int_0^\infty [G_0(x, x, E) - G_1(x, x, E)] dx = \frac{\delta_2}{2(\kappa^2 - ia\kappa)} - \frac{\delta_1}{\kappa^2 + a^2} \tag{27}$$

where $E = \kappa^2$ and $\alpha = -a^2$; for case (i), $\delta_1 = \delta_2 = 1$ and for case (iii) $\delta_1 = 0$, $\delta_2 = -1$.

Proof. The proof is based on the fact that for such potentials the left solution goes to zero like x when $x \rightarrow 0$ and the right solution has the asymptotics $f_{r0} \sim \exp(i\kappa x)$, $E = \kappa^2$, $\text{Im } \kappa > 0$ (see, e.g., [8]). It follows from here that when E is not a spectral point, the left solution has a growing asymptotics at infinity, $f_{l0} \sim -\frac{W_0}{2i\kappa} \exp(-i\kappa x)$, $x \rightarrow \infty$, where W_0 is the Wronskian of f_{r0} and f_{l0} . It is not possible to create a new bound state in this case since the Schrödinger equation with such a potential has no solutions going to infinity as x approaches the origin. \square

Example 1. Free motion on the line, $V_0(x) = 0$, $x \in \mathbb{R}$.

The Green function is

$$G_0(x, y, E) = \frac{i}{2\kappa} e^{i\kappa|x-y|} \quad \text{Im } \kappa > 0 \quad E = \kappa^2.$$

The choice $u = \cosh(ax)$, $\alpha = -a^2$, leads to the one-soliton potential $V_1 = -2a^2 \text{sech}^2(ax)$ with the Green function

$$G_1(x, y, E) = i f_\kappa(x) f_{-\kappa}(y) / [2\kappa(\kappa^2 + a^2)] \quad x \leq y$$

where $f_\kappa(x) = \exp(-i\kappa x)(i\kappa + a \tanh ax)$, which clearly has a pole at the ground-state energy $E = -a^2$. The residue at the pole is $\varphi_0(x)\varphi_0(y)$ where $\varphi_0(x) = \sqrt{a/2} \text{sech}(ax)$ which is just the ground state of the one-soliton potential.

Continuous spectrum eigenfunctions of h_0 , $\psi_k(x) = 1/\sqrt{2\pi} \exp(ikx)$ are transformed into continuous spectrum eigenfunctions for h_1 , $\xi_k(x) = [-i\kappa + \tanh(ax)] \exp(ikx) / \sqrt{2\pi(\kappa^2 + a^2)}$. The direct calculation of the function $P(k)$ given in (25) gives exactly the result (26).

Example 2. Free motion on a half line with zero angular momentum, $V_0(x) = 0$, $x \in \mathbb{R}^+$.

The Green function is

$$G_0(x, y, E) = \frac{1}{\kappa} \sin(\kappa x) \exp(i\kappa y) \quad E = \kappa^2 \quad \text{Im } \kappa > 0 \quad x \leq y.$$

It must be noted that the use of the transformation function $u = \cosh(ax)$ gives the same one-soliton potential as in the previous example which when considered on a half line has only a continuous spectrum but its continuous spectrum eigenfunctions cannot be obtained by applying the transformation operator L to the free particle eigenfunctions since it does not preserve the zero boundary condition at the origin. Another choice $u = \sinh(ax)$ creates the potential $V_1 = 2a^2 \text{csch}^2(ax)$ which is singular at the origin and has only a continuous spectrum and which may be obtained with the help of the SUSY transformation from the free particle Hamiltonian. Its Green function is

$$G_1(x, y, E) = \frac{i e^{i\kappa y}}{\kappa(\kappa^2 + a^2)} [\kappa + ia \text{cth}(ay)] [\kappa \cos(\kappa x) - a \text{cth}(ax) \sin(\kappa x)]$$

$$E = \kappa^2 \quad \text{Im } \kappa > 0.$$

It is not difficult to see that despite the presence of the denominator, G_1 is regular for all $\kappa \neq 0$ including the point $\kappa = ia$ and discontinuous along the positive real axis in the complex E -plane. The jump across this cut is proportional to the product of continuous spectrum eigenfunctions of h_1 which are given by

$$\xi_k = \sqrt{\frac{2}{\pi(k^2 + a^2)}} [-k \cos(kx) + a \operatorname{cth}(ax) \sin(kx)].$$

In contrast to the previous example, the difference $\psi_k^2 - \xi_k^2$ now oscillates when $x \rightarrow \infty$, the Riemann integral of this difference over the space variable is divergent and the interchange of the integrals over k and x is impossible. Nevertheless, if one assumes that the improper integral over x is a limit of a proper integral one can write

$$\lim_{A \rightarrow \infty} \int_0^A \frac{P(k, A) dk}{k^2 - E} = R(E) \quad R(E) = \frac{1}{2(ia\kappa - \kappa^2)} \quad (28)$$

where $E = \kappa^2$, $\alpha = -a^2$ and $P(k, A) = \int_0^A [\psi_k^2(x) - \chi_k^2(x)] dx$.

In our example, the function $P(k, A)$ is given by

$$P(k, A) = \frac{2a \operatorname{cth}(aA) \sin^2(kA) - k \sin(2Ak)}{\pi(k^2 + a^2)}$$

which when substituted into the left-hand side of (28) gives exactly the function $R(E)$. So this example shows that in contrast to the previous example where the difference of normalizations is a finite quantity, here this value is undetermined. Nevertheless, the contribution to the trace of the difference $G_0 - G_1$ from the continuous spectra of h_0 and h_1 is well defined.

5. Conclusions

In this paper, we have studied the relation between the Green functions corresponding to two Hamiltonians which are SUSY partners. We have shown that it is possible to establish a relation between the traces of the Green functions for the two partner Hamiltonians for the cases of deletion of the ground state, the addition of a new ground state and when the two Hamiltonians are isospectral. The formulae derived in this paper are valid for the general case of Hamiltonians having both discrete and continuous spectra. Our results show that when a continuous spectrum is present, each of the traces of the Green functions for the SUSY partners may diverge but the difference between the traces can be finite. We have illustrated our results by considering the case of the free motion on the full line and the case of the free motion of a particle with zero angular momentum on the half line.

Finally, we would like to note that the difference of the traces of the Green functions of the two SUSY partner Hamiltonians appears as the trace (actually super-trace) of the Green function of the supersymmetric Schrödinger equation (supersymmetric Green function). Thus, our results reveal the possibility of divergence of the component traces of the supersymmetric Green function while its super-trace remains finite.

Acknowledgments

The work of BFS is partially supported by the President Grant of Russia 1743.2003.2 and the Spanish MCYT and European FEDER grant BFM2002-03773.

References

- [1] Sukumar C V 2004 *J. Phys. A: Math. Gen.* **37** 10287
- [2] Samsonov B F and Pupasov A M *Russ. Phys. J.* (submitted)
- [3] Morse P and Feshbach H 1953 *Methods of Theoretical Physics* vol 1 (New York: McGraw-Hill)
- [4] Levitan B M and Sargsyan I S 1975 *Introduction to Spectral Theory: Selfadjoint Ordinary Differential Operators* (Providence, RI: American Mathematical Society)
- [5] Berezin F A and Shubin M A 1991 *The Schrödinger Equation* (Dordrecht: Kluwer)
- [6] Sukumar C V 1985 *J. Phys. A: Math. Gen.* **18** 2917
- [7] Levitan B M 1984 *Inverse Sturm–Liouville Problems* (Moscow: Nauka)
- [8] Faddeev L D 1963 *J. Math. Phys.* **4** 72